On the Kowalevski-Goryachev-Chaplygin gyrostat

2002 J. Phys. A: Math. Gen. 35 L309

(http://iopscience.iop.org/0305-4470/35/22/103)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) L309–L318 PII: S0305-4470(02)34960-6

LETTER TO THE EDITOR

On the Kowalevski–Goryachev–Chaplygin gyrostat

A V Tsiganov

Department of Mathematical and Computational Physics, Institute of Physics, St Petersburg University, 198 904, St Petersburg, Russia

E-mail: tsiganov@mph.phys.spbu.ru

Received 15 March 2002, in final form 30 April 2002
Published 24 May 2002
Online at stacks.iop.org/JPhysA/35/L309

Abstract
The invariant separated variables are constructed for the Kowalevski–Goryachev–Chaplygin gyrostat using decomposition of the corresponding Lagrangian submanifold on a symmetric product of two copies of single spectral curve of the 2×2 Lax matrix. A generalization to the bundle of the Poisson brackets is discussed.

PACS numbers: 02.30.Ik, 02.40.−k

1. Introduction

In this letter, we consider the Kowalevski–Goryachev–Chaplygin gyrostat with the following Hamilton function [1]:

$$H = J_1^2 + J_2^2 + 2J_3^2 + 2\rho J_3 + c_1x_1 + c_2x_2 + c_3(x_1^2 - x_2^2) + c_4x_1x_2 + \frac{\delta}{x_3^2} \qquad c_1, c_2, c_3, c_4, \rho, \delta \in \mathbb{R}. \tag{1.1}$$

The phase space consists of the variables x_i and J_i. The position of a rigid body is fixed by the components x_i of the Poisson vector, which are cosines between the axes of the body frame and the field up to a constant. The J_i are components of the angular momentum in the body-fixed frame of reference.

Below we shall identify the phase space with union of the coadjoint orbits (symplectic leaves) of the Euclidean motion group $E(3)$ in $e^*(3)$. In this case J_i, x_i, $i = 1, 2, 3$, are coordinates on the dual Lie algebra $e^*(3)$ with the standard Lie–Poisson brackets

$$\{J_i, J_j\} = \varepsilon_{ijk}J_k, \qquad \{J_i, x_j\} = \varepsilon_{ijk}x_k, \qquad \{x_i, x_j\} = 0, \qquad i, j, k = 1, 2, 3, \tag{1.2}$$

where ε_{ijk} is the standard totally skew-symmetric tensor. The generic symplectic leaves

$$\mathcal{E}_{ab} : \{(x, J) : \boldsymbol{C}_1 = a, \boldsymbol{C}_2 = b\}$$

are four-dimensional symplectic manifolds specified by values of two Casimir elements

$$\boldsymbol{C}_1 = \sum_{i=1}^{3} x_i^2 = a, \qquad \boldsymbol{C}_2 = \sum_{i=1}^{3} x_i J_i = b. \tag{1.3}$$

The symplectic $\mathcal{E}_{ab}$ leaves are topologically equivalent to the cotangent bundle of the sphere T^*S^2.

The Hamilton function (1.1) determines the dynamical system on $\mathcal{E}_{ab}$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \{H, .\} \tag{1.4}$$

which is integrable by Liouville on the union of non-generic orbits $\mathcal{E}_a$ defined by the zero value of the second Casimir function

$$\boldsymbol{C}_2 = \sum_{i=1}^{3} x_i J_i = 0. \tag{1.5}$$

The corresponding additional integral is given by

$$\begin{aligned} K = {} & \left(J_1^2 + J_2^2 - 2\rho J_3 - \rho^2 + c_1 x_1 + c_2 x_2 + \frac{\delta}{x_3^2} \right)^2 \\ & + 2c_3((x_1 - x_2)\rho + x_3(J_1 - J_2))((x_1 + x_2)\rho + x_3(J_1 + J_2)) \\ & + 2c_4(\rho x_2 + x_3 J_2)(\rho x_1 + x_3 J_1) + 4(J_3 + \rho)((\rho x_1 + x_3 J_1)c_1 + c_2(\rho x_2 + x_3 J_2)) \\ & - 2c_1 c_2 x_1 x_2 - \frac{c_1^2 - c_2^2}{2}(x_1^2 - x_2^2) - \frac{c_1^2 + c_2^2}{2} x_3^2 \\ & + (2(c_2 x_2 - c_1 x_1)c_3 - c_4(c_1 x_2 + c_2 x_1))x_3^2 + \left(c_3^2 + \frac{c_4^2}{4} \right) x_3^4. \end{aligned} \tag{1.6}$$

The level surface of integrals of motion

$$\mathcal{C}^{(2)} : ((x, J) \in \mathcal{E}_a : \ H = \alpha_1, K = \alpha_2) \tag{1.7}$$

is a two-dimensional Lagrangian submanifold, which is a graph of the action function $\mathcal{S}$ [3].

In the separation of variables method we suppose that the complete integral $\mathcal{S}$ of the corresponding Hamilton–Jacobi equation has an additive form

$$\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2. \tag{1.8}$$

This means that the Lagrangian submanifold $\mathcal{C}^{(2)}$ (1.7) is realized as a product

$$\mathcal{C}^{(2)} = \mathcal{C}_1 \times \mathcal{C}_2$$

of two plane curves $\mathcal{C}_j$, which are graphs of $\mathrm{d}\mathcal{S}_j$. These curves are defined by some equations

$$\mathcal{C}_j : \Phi_j(\lambda_j, \mu_j, \alpha_1, \alpha_2, a) = 0. \tag{1.9}$$

Here μ_j and λ_j are coordinates on the plane related by the algebraic relation (1.9).

Then we are looking for a special canonical transformation of variables $(x, J) \mapsto (p, q)$ such that each function $\mathcal{S}_j$ in (1.8) becomes a function on a single coordinate q_j only. In this case separated variables (p, q) satisfy one-dimensional separated equations

$$p_j = \frac{\partial \mathcal{S}_j(q_j, \alpha_1, \alpha_2, a)}{\partial q_j}. \tag{1.10}$$

Since functions $\mathcal{S}_j$ are generating functions of the one-dimensional Lagrangian submanifolds $\mathcal{C}_j$, the pairs of separated variables lie on the curves $\mathcal{C}_j$

$$\Phi_j(p_j, q_j, H(p, q), K(p, q), \boldsymbol{C}_1(p, q)) = 0. \tag{1.11}$$

These separated equations are obtained from (1.9) after substitution of the integrals of motion as functions of the separated variables instead of their values α_j and the separated variables instead of plane coordinates $\mu = \phi(P_i, Q_i)$ and $\lambda = \psi(P_i, Q_i)$, which become functions on the phase space.

Thus, in the separation of variables method we have to realize the Lagrangian submanifold $\mathcal{C}^{(2)}$ as a product of one-dimensional Lagrangian submanifolds $\mathcal{C}_j$ and have to determine the corresponding canonical separated variables p_i, q_i.

In the special case $c_1 = c_2 = c_4 = \rho = \delta = 0$ the dynamical equations (1.4) are the Kirchhoff equations. The corresponding separated variables have been invented by Chaplygin [2]. In this case, the Lagrangian submanifold is a level surface of new functions $I_{1,2}$ on the initial integrals H and K

$$\tilde{\mathcal{C}}^{(2)} : \{(x, J) \in \mathcal{E}_a : \ I_{1,2} = H \pm \sqrt{K} = \alpha_{1,2}\}.$$

It may be realized as a product of two plane curves $\tilde{\mathcal{C}}_j$ defined by

$$\tilde{\mathcal{C}}_{1,2} : \Phi_{1,2}(\mu, \lambda) = \mu^2 - \frac{c_3 a\lambda + \alpha_{1,2}}{2(\lambda^2 - a)} = 0. \tag{1.12}$$

The corresponding separated coordinates $q_{1,2}$ are zeros of the polynomial

$$\lambda^2 - 2\lambda\left(\frac{J_1^2 + J_2^2}{c_3 x_3^2}\right) + \frac{(J_1^2 + J_2^2)^2 - K}{c_3^2 x_3^4} = 0, \tag{1.13}$$

whereas the conjugated momenta may be determined from equations (1.12) by $\lambda = q_{1,2}$ and $\mu = p_{1,2}$.

If $c_2 = c_3 = c_4 = \delta = 0$ the equations of motion (1.4) are the Euler–Poisson equations, which were rewritten in the Lax form by Reyman and Semenov-Tian Shansky [4]. In this case the Lagrangian submanifold $\mathcal{C}^{(2)}$ (1.7) may be realized as a symmetric product of two copies of the spectral curve $\hat{\mathcal{C}}$ of the Lax matrix $L(\lambda)$. The corresponding characteristic equation is given by

$$\begin{aligned}\hat{\mathcal{C}} : \Phi(\mu, \lambda) = \det(L(\lambda) - \mu)) &= \mu^4 - \mu^2(2c_1^2\lambda^2 + 4(\alpha_1 + \rho^2)\lambda - 4) \\ &\quad + (c_1^4\lambda^4 - 4c_1^2(\alpha_1 + \rho^2)\lambda^3 - (4\alpha_2 - 2c_1^2)\lambda^2 - 8\rho^2\lambda) = 0.\end{aligned} \tag{1.14}$$

In the framework of the Sklyanin method [5] the corresponding separated variables $q_{1,2}$ were constructed in [6]. They are poles of the Baker–Akhiezer function on $\hat{\mathcal{C}}$ with standard normalization. According to [6], we can select two zeros of the polynomial

$$(\lambda^2 - q_1^2)(\lambda^2 - q_2^2) = \lambda^4 + B_1\lambda^2 + B_0 = 0 \tag{1.15}$$

as the separated variables $q_{1,2}$. Here

$$B_1 = \frac{c_1^2 x_-^2 J_-^2 - c_1(2J_-(J_3 + \rho) - c_1 x_3)(2x_3 J_-^2 - x_-(2J_- J_3 + c_1 x_3))}{4(J_-^2 - c_1 x_-)J_-^2},$$

$$B_0 = \frac{c_1^2(2J_-(J_3 + \rho) - c_1 x_3)^2}{16(J_-^2 - c_1 x_-)J_-^2}, \qquad \text{and} \qquad x_\pm = x_1 \pm \mathrm{i}x_2, \qquad J_\pm = J_1 \pm \mathrm{i}J_2.$$

Variables $q_{1,2}$ together with canonically conjugated momenta $p_{1,2}$ lie on a spectral curve (1.14).

However, these separated variables $q_{1,2}$ (1.15) are complex rational functions on real physical variables x and J. A similar situation arises for the Lagrange top as well. If the Lagrangian submanifold is a product of two *different* curves, then the corresponding separated variables are real functions on physical variables (see [7]). If the Lagrangian submanifold is a symmetric product of two copies of *single* spectral curve of the Lax matrix, then the separated variables are complex functions of physical variables. In the framework of the Sklyanin method and bi-Hamiltonian geometry these separated variables were constructed in [8].

The aim of this letter is to present the separated variables for the Kowalevski–Goryachev–Chaplygin gyrostat (1.1) with six arbitrary parameters c_i, ρ and δ. We realize Lagrangian submanifold (1.7) as a symmetric product of two copies of the spectral curve of 2×2 Lax matrix proposed in [9] by $\rho = 0$ and generalized in [10] by $\rho \neq 0$. As above, the associated separated variables are complex functions on initial physical variables x and J. A generalization of these results to the bundle of the Poisson brackets is obtained.

2. The Lax matrix

According to [9, 10], let us start with the 2×2 Lax matrix for the symmetric Neumann system

$$T(\lambda) = \begin{pmatrix} \lambda^2 - 2J_3\lambda - J_1^2 - J_2^2 - \frac{\delta}{x_3^2} & \lambda(\mathrm{i}x_1 + x_2) - x_3(\mathrm{i}J_1 + J_2) \\ \lambda(\mathrm{i}x_1 - x_2) - x_3(\mathrm{i}J_1 - J_2) & x_3^2 \end{pmatrix}, \qquad (2.1)$$

which is a matrix polynomial on spectral parameter λ.

Proposition 1. [9, 10] *Matrix $T(\lambda)$ (2.1) defines representation of the Sklyanin algebra*

$$\{\overset{1}{T}(\lambda), \overset{2}{T}(\nu)\} = [r(\lambda - \nu), \overset{1}{T}(\lambda)\,\overset{2}{T}(\nu)], \qquad (2.2)$$

on the symplectic leaves $\mathcal{E}_a$ (1.5).

Here we use the standard notations $\overset{1}{T}(\lambda) = T(\lambda) \otimes Id$, $\overset{2}{T}(\nu) = Id \otimes T(\nu)$,

$$r(\lambda - \nu) = \frac{\mathrm{i}}{\lambda - \nu}\Pi, \qquad \text{and} \qquad \Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (2.3)$$

Notion of this representation of the Sklyanin algebra $T(\lambda)$ (2.1) allows us to construct the Lax matrices for the Goryachev–Chaplygin top [10], its generalization [11] and the Lax matrices for the Kowalevski–Goryachev–Chaplygin top [9] and gyrostat [10].

Recall that the main property of the Sklyanin algebra (2.2) is that for *any* numerical matrix $\mathcal{K}$ coefficients of the trace of matrix $\mathcal{K}T(\lambda)$ give rise to the commutative subalgebra

$$\{\mathrm{tr}\,\mathcal{K}T(\lambda), \mathrm{tr}\,\mathcal{K}T(\nu)\} = 0.$$

All the generators of this subalgebra are linear polynomials on coefficients of entries $T_{ij}(\lambda)$, which are interpreted as integrals of motion for integrable system associated with the matrix $T(\lambda)$.

According to [12], we can construct another commutative subalgebra generated by *quadratic* polynomials on coefficients of $T_{ij}(\lambda)$, which are integrals of motion for another integrable system associated with the same matrix $T(\lambda)$. Recall that, if $\mathcal{K}_\pm(\lambda)$ are solutions of the reflection equation

$$\{\overset{1}{\mathcal{K}}(\lambda), \overset{2}{\mathcal{K}}(\nu)\} = [r(\lambda - \nu), \overset{1}{\mathcal{K}}(\lambda)\overset{2}{\mathcal{K}}(\nu)] + \overset{1}{\mathcal{K}}(\lambda)r(\lambda + \nu)\overset{2}{\mathcal{K}}(\nu) - \overset{2}{\mathcal{K}}(\nu)r(\lambda + \nu)\overset{1}{\mathcal{K}}(\lambda), \qquad (2.4)$$

then coefficients of the trace of the Lax matrix

$$L(\lambda) = \mathcal{K}_-(\lambda)\, T(\lambda - \rho)\, \mathcal{K}_+(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T^{\mathrm{t}}(-\lambda - \rho) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (2.5)$$

give rise to the commutative subalgebra

$$\{\mathrm{tr}\, L(\lambda), \mathrm{tr}\, L(\nu)\} = 0.$$

In (2.5) the superscript t stands for matrix transposition; the matrix $T(\lambda)$ satisfies (2.2) and commutes with $\mathcal{K}(\lambda)$.

In particular, using this general approach we can construct the Lax matrix for the Kowalevski–Goryachev–Chaplygin gyrostat (1.1) [9, 10]. Namely, substituting the matrix $T(\lambda)$ (2.1) and two special numerical solutions of the reflection equations (2.4)

$$\mathcal{K}_+ = \begin{pmatrix} a_1\lambda + d_1 & \lambda \\ 0 & -a_1\lambda + d_1 \end{pmatrix}, \qquad \mathcal{K}_- = \begin{pmatrix} a_2\lambda + d_2 & 0 \\ \lambda & -a_2\lambda + d_2 \end{pmatrix} \tag{2.6}$$

depending on arbitrary parameters $a_{1,2}, d_{1,2}$ into the definition (2.5) one obtains the desired Lax matrix. The spectral curve of this Lax matrix $L(\lambda)$ (2.5) is defined by the characteristic equation

$$\begin{aligned} \mathcal{C} : \Phi(\mu,\lambda) = \det(L(\lambda) - \mu) &= \mu^2 + \mu(\lambda^6 - 2\tilde{H}\lambda^4 + \tilde{K}\lambda^2 + 2d_1d_2(a\rho^2 - \delta)) \\ &\quad + (a_1^2\lambda^2 - d_1^2)(a_2^2\lambda^2 - d_2^2)(a(\rho-\lambda)^2 - \delta)(a(\rho+\lambda)^2 - \delta), \end{aligned} \tag{2.7}$$

where $a = \boldsymbol{C}_1$ (1.3) is a Casimir element. Integrals of motion

$$\begin{aligned} \tilde{H} = J_1^2 + J_2^2 + 2J_3^2 + 2\rho J_3 + \rho^2 - a_1a_2a - \mathrm{i}(d_1 + d_2 - (a_1 + a_2)(2J_3 + \rho))x_1 \\ - (d_1 - d_2 - (a_1 - a_2)(2J_3 + \rho))x_2 + (\mathrm{i}(a_1 + a_2)J_1 + (a_1 - a_2)J_2)x_3 + \frac{\delta}{x_3^2} \end{aligned}$$

and $\tilde{K}$ in (2.7) coincide with the previous integrals H (1.1) and K (1.6) up to the Casimir function and after canonical transformation of variables

$$J \to J + Ux, \qquad U = \begin{pmatrix} 0 & 0 & \mathrm{i}\beta_+ \\ 0 & 0 & \beta_- \\ -\mathrm{i}\beta_+ & -\beta_- & 0 \end{pmatrix}, \qquad \beta_\pm = \frac{a_1 \pm a_2}{2}$$

and exchange of parameters

$$a_1^2 = \left(c_3 + \frac{\mathrm{i}c_4}{2}\right), \qquad a_2^2 = \left(c_3 - \frac{\mathrm{i}c_4}{2}\right), \qquad d_1 = \frac{\mathrm{i}c_1 - c_2}{2}, \qquad d_2 = \frac{\mathrm{i}c_1 + c_2}{2}.$$

Thus, we obtain the 2×2 Lax matrix for the Kowalevski–Goryachev–Chaplygin gyrostat (1.1) and realize the Lagrangian submanifold $\mathcal{C}^{(2)}$ (1.7) as a product of two copies of the spectral curve $\mathcal{C}$ (2.7).

Note, in contrast with our previous papers [9, 10], that here we use other matrices $\mathcal{K}_\pm$ and, therefore, another decomposition of the Lagrangian submanifold $\mathcal{C}^{(2)}$ (1.7).

Using this 2×2 Lax matrix and the machinery of finite-band integration theory [13] we could obtain explicit expressions for the solutions of the Kowalevski–Goryachev–Chaplygin gyrostat. However, it is impossible now to find a precise quantum analogue of these classical constructions. Therefore, below we shall consider the separation of variables method, which is another universal method of solving completely integrable classical and quantum models [5].

3. The separated variables

The separated variables are poles of the properly normalized Baker–Akhiezer function on the spectral curve $\mathcal{C}$ (2.7). Using the known 2×2 Lax matrix (2.5) these variables may be easy found in the framework of the Sklyanin method [5].

However, one could construct the same decomposition of the Lagrangian submanifold $\mathcal{C}^{(2)}$ (1.7) and obtain the equation of the curve $\mathcal{C}$ (2.7) using another method. For instance, in order to obtain the equation of the plane curve $\mathcal{C}$ one can apply the singular analysis of equations of motion or bi-Hamiltonian geometry. So, it is interesting to construct the separated variables directly from the definition of the plane curve $\mathcal{C}$ (2.7).

The Poisson manifold $e^*(3)$ is a regular transversally constant manifold. There exist two vector fields Z_a and Z_b which are transversal to the symplectic leaves $\mathcal{E}_{ab}$ and are symmetries of the Poisson tensor. So, locally $\mathcal{M}$ is a product of the symplectic leaf and of the Abelian group G generated by the vector fields $Z_{a,b}$.

The integrable system on symplectic leaves $\mathcal{E}_{ab}$ consists of an integrable Lagrangian foliation (distribution). For instance, in our case fibres of this foliation are equal to $\mathcal{C}^{(2)} = \mathcal{C}\times\mathcal{C}$, where $\mathcal{C}$ is defined by (2.7). The vector fields $Z_{a,b}$ are transversal to all the Lagrangian fibres (Liouville tori) as the vector fields associated with the action variables. If connections of the Lagrangian and symplectic foliations are compatible, then the integrable system (Lagrangian foliation) is invariant with respect to action of the Abelian group G.

It is natural to suppose that the desired separated variables for an invariant integrable system are invariant with respect to action of the Abelian group G as well. If this is true, then the separated variables are solutions of the following system of equations [14]:

$$\Phi(\mu,\lambda,H,K,a)|_{\mathcal{E}_{ab}} = 0, \qquad Z_{a,b}\Phi(\mu,\lambda,H,K,a)|_{\mathcal{E}_{ab}} = 0. \tag{3.1}$$

Here the function $\Phi(\mu,\lambda,H,K,a)$ defines the algebraic curve $\mathcal{C}$ (2.7) and the vector fields $Z_{a,b}$ locally are equal to $Z_a = \partial/\partial a$.

In our case $b = 0$ (1.5) and we have only one field Z_a associated with the canonical transformation $x \to ax$, which describes the symmetry of the Poisson tensor. This canonical transformation generates a change of parameters in the Hamilton function H (1.1)

$$c_{1,2} \to ac_{1,2}, \qquad c_{3,4} \to a^2 c_{3,4}.$$

This allows us to enlarge the system of equations (3.1) using a few vector fields $Z_{c_j} = \partial/\partial c_j$ instead of a single field Z_a, as for the Neumann system and as for the Goryachev–Chaplygin top [14].

Proposition 2. *The separated variables associated with the curve $\mathcal{C}$ (2.7) are solutions of one of the following systems of equations for $j = 1, 2$:*

$$\begin{aligned} &\Phi(\mu,\lambda,H,K,a) = 0, \qquad \frac{\partial}{\partial a_j}\Phi(\mu,\lambda,H,K,a) = 0,\\ &\frac{\partial}{\partial d_j}\Phi(\mu,\lambda,H,K,a) = 0 \end{aligned} \tag{3.2}$$

or

$$\begin{aligned} &\Phi(\mu,\lambda,H,K,a) = 0, \qquad \Phi(\mu,\lambda,H,K,a)|_{a_j=0} = 0,\\ &\Phi(\mu,\lambda,H,K,a)|_{d_j=0} = 0. \end{aligned} \tag{3.3}$$

Proof. This proposition may be proved by straightforward calculations. For brevity, to prove this proposition we shall use the known Lax representation $L(\lambda)$ (2.5). □

Let us begin with the case $j = 2$. Using matrix entries of the subsidiary matrix

$$\mathcal{T}(\lambda) = T(\lambda-\rho)\,\mathcal{K}_+(\lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T^{\mathrm{t}}(-\lambda-\rho)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.4}$$

which satisfy the reflection equation (2.4), one obtains

$$\Phi(\mu,\lambda) = \mu^2 + ((-a_2\lambda - d_2)\mathcal{T}_{11} + (a_2\lambda - d_2)\mathcal{T}_{22} - \lambda\mathcal{T}_{12})\mu - (a_2^2\lambda^2 - d_2^2)\mathrm{Det}(\mathcal{T}).$$

Substituting this function $\Phi(\mu,\lambda)$ into (3.2) or (3.3) and eliminating μ, we obtain an equation for the definition of the separated coordinates $q_{1,2}$ as zeros of the following polynomial:

$$\mathcal{T}_{12}(\lambda) = \lambda(\lambda^2 - q_1^2)(\lambda^2 - q_2^2) = 0. \tag{3.5}$$

The remaining equation defines μ_k as a function of coordinates q_k and momenta p_k

$$\mu_k(p_k,q_k) = (a_2q_k + d_2)\mathcal{T}_{11}(q_k). \tag{3.6}$$

The variables p_k and q_j are required to be canonical

$$\{q_1, q_2\} = \{p_1, p_2\} = 0, \qquad \{p_k, q_j\} = \delta_{kj}.$$

Putting together this condition and equations

$$\{q_1, q_2\} = \{\mu_1, \mu_2\} = 0, \qquad \{q_j, \mu_k\} = -\mathrm{i}\mu_k\delta_{jk},$$

which follows from the definitions (3.5) and (3.6), one obtains

$$p_k = -\mathrm{i}\ln\mathcal{T}_{11}(q_k) = -\mathrm{i}(\ln\mu_k - \ln(a_2q_k + d_2)). \tag{3.7}$$

To end the proof it is sufficient now to notice that by definition (3.2) canonical variables $p_{1,2}$ and $q_{1,2}$ lie on the curve $\mathcal{C}$ and, therefore, they are separated variables.

Substituting $\lambda = q_j$ and μ_k into equation (2.7) one obtains the corresponding separated equations (1.11). So, the system of equations of motion reduces to quadratures on the Jacobian of a hyperelliptic curve (2.7) of genus $g = 5$. For $\rho = 0$, i.e. without a gyrostatic term, the genus of the curve is reduced to $g = 2$.

If $j = 1$ we can exchange factors in (2.5) and introduce another Lax matrix associated with the same spectral curve $\mathcal{C}$ (2.7)

$$\tilde{L}(\lambda) = \tilde{\mathcal{T}}\mathcal{K}_+,$$

where

$$\tilde{\mathcal{T}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T^{\mathrm{t}}(-\lambda-\rho) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{K}_-(\lambda)T(\lambda-\rho).$$

By the same methods it can be shown that the separated coordinates $\tilde{q}_{1,2}$ are zeros of the following polynomial:

$$\tilde{\mathcal{T}}_{21}(\lambda) = \lambda(\lambda^2 - \tilde{q}_1^2)(\lambda^2 - \tilde{q}_2^2) = 0 \tag{3.8}$$

and the conjugated momenta read as

$$\tilde{p}_k = -\mathrm{i}\ln\tilde{\mathcal{T}}_{11}(\tilde{q}_k).$$

As above these canonical variables $\tilde{p}_{1,2}$ and $\tilde{q}_{1,2}$ lie on the curve $\mathcal{C}$ and, therefore, they are separated variables.

Remark. The pairs of variables $q_{1,2}$ and $\tilde{q}_1^2$ are related by canonical transformation of the physical variables $x_2 \to -x_2$, $J_2 \to -J_2$ and by flip of parameters $a_1 \leftrightarrow a_2$, $d_1 \leftrightarrow d_2$. The existence of two pairs of separated variables is associated with the invariance of the Sklyanin brackets (2.2) with respect to a matrix transposition $T \to T^{\mathrm{t}}$.

Equation (3.5) has the following form:

$$\lambda^4 + 2(H - \mathrm{i}v - \rho^2)\lambda^2 + 2\mathrm{i}u + 2d_1d_2a - K = 0 \tag{3.9}$$

where

$$v = a_2x_3J_+ - (a_2(2J_3+\rho) - d_2 - \mathrm{i}a_2^2x_+)x_+$$

$$\begin{aligned} u = {} & (J_+^2 + 2\mathrm{i}d_1x_+ + a_1^2x_3^2)(a_2x_3J_- - d_2x_- + \mathrm{i}a_2^2x_3^2) \\ & + \left(\rho(2\mathrm{i}a_2^2x_+ - a_2(2J_3+\rho) + 2d_2) + \frac{a_2\delta}{x_3^2}\right)(x_3J_+ + \rho x_+) \\ & - \mathrm{i}\rho^2(a_2^2x_+ - \mathrm{i}d_2)x_+ + (a_2x_3^2(-2\mathrm{i}d_1 + a_1^2x_-) + a_2x_+J_+J_-)\rho + \frac{d_2\delta x_+}{x_3^2}, \end{aligned}$$

and

$$x_\pm = x_1 \pm \mathrm{i}x_2, \qquad J_\pm = J_1 \pm \mathrm{i}J_2.$$

The separated coordinates $q_{1,2}$ are zeros of the polynomial (3.9).

In order to compare our results with known ones (1.13) and (1.15) we present a definition of the separated coordinates (3.9) for these special cases explicitly. For $c_1 = c_2 = c_4 = \rho = \delta = 0$ polynomial (3.9) reads as

$$\lambda^4 + 2(H + a_2^2 x_+^2 - \mathrm{i}a_2(x_3 J_+ - 2x_+ J_3))\lambda^2 + 2\mathrm{i}a_2 x_3 (J_+^2 + a_2^2 x_3^2)(J_- + \mathrm{i}a_2 x_3) - K = 0,$$

and for $c_2 = c_3 = c_4 = \delta = 0$ it has the form

$$\lambda^4 + 2(H - \mathrm{i}d_2 x_+ - \rho^2)\lambda^2 + 2\mathrm{i}d_2(\rho^2 x_+ + J_+(2\rho x_3 - x_- J_+)) + 2d_2^2 a - K = 0.$$

The main difference from the previous constructions [2] and [6] is that coefficients of the equation (3.9) are polynomials on initial physical variables. This allows us to construct the quantum counterpart of equation (3.9) using the quantum matrix $T(\lambda)$ proposed in [9].

The separated variables (1.13) and new variables (3.9) proposed in [2] are associated with the different Lagrangian submanifolds defined by $H \pm \sqrt{K} = \alpha_{1,2}$ and $H = \alpha_1$, $K = \alpha_2$, respectively. Henceforth, there are two different separations of variables associated with two level surfaces of integrals with different topology.

On the other hand, the separated variables (1.13) [2] and new variables (3.9) proposed in [6] are associated with the different decomposition of the common Lagrangian submanifold defined by $H = \alpha_1$, $K = \alpha_2$. Since the motion linearizes on the Jacobians of the curves (1.14) and (2.7) which are isogeneous to one another [15], we can map one set of the separated equations (1.14) into another one (2.7) using an algebro-geometric approach [15]. It remains an interesting problem to study the induced interrelations between the corresponding separated variables.

4. Generalization

Let $J_i, y_i, i = 1, 2, 3$, be coordinates on the six-dimensional twisted Kac–Moody algebra $\mathfrak{g}_\varkappa$ with the following Lie–Poisson brackets:

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \qquad \{J_i, y_j\} = \varepsilon_{ijk} y_k, \qquad \{y_i, y_j\} = \varkappa^2 \varepsilon_{ijk} J_k, \tag{4.1}$$

where $\varkappa^2$ is an arbitrary parameter. The generic symplectic leaves

$$\mathcal{O}_{ab} : \{(x, J) : \tilde{C}_1 = a, \tilde{C}_2 = b\}$$

are specified by the fixed values of two Casimir elements

$$\tilde{C}_1 = \sum_{i=1}^{3} x_i^2 + \varkappa^2 \sum_{i=1}^{3} J_i^2, \qquad \tilde{C}_2 = \sum_{i=1}^{3} y_i J_i. \tag{4.2}$$

By $\varkappa = 0$ one obtains the Euclidean $\mathfrak{g}_0 \simeq e(3)$ algebra; by $\varkappa = 1, \sqrt{-1}$ one obtains another semisimple algebras $\mathfrak{g}_1 \simeq \mathrm{o}(4)$ and $\mathfrak{g}_{\sqrt{-1}} \simeq \mathrm{o}(3, 1)$, respectively.

Proposition 3. *If $C_2 = \tilde{C}_2 = 0$ the equation*

$$\frac{\boldsymbol{y}}{|\boldsymbol{y}|} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \tag{4.3}$$

describes the isomorphism of the non-generic orbits $\mathcal{O}_a$ and $\mathcal{E}_a$.

Proof. Straightforward calculations show that the scaling transformation

$$x_i = \frac{a y_i}{\sqrt{y_1^2 + y_2^2 + y_3^2}}$$

and the inverse transformation

$$y_i = x_i \sqrt{\frac{a - \varkappa^2(J_1^2 + J_2^2 + J_3^2)}{x_1^2 + x_2^2 + x_3^2}}$$

relate brackets (4.1) to brackets (1.2) by $\boldsymbol{C}_2 = \tilde{\boldsymbol{C}}_2 = 0$. □

Using these maps we can construct new integrable systems on the bundle of the Poisson brackets (4.1). For instance, starting with the Hamilton function H (1.1) one obtains the new Hamiltonian

$$H_\varkappa = J_1^2 + J_2^2 + 2J_3^2 + 2\rho J_3 + \frac{c_1 y_1 + c_2 y_2}{|y|} + \frac{c_3(y_1^2 - y_2^2) + c_4 y_1 y_2}{|y|^2} + \frac{\delta(y_1^2 + y_2^2)}{y_3^2}.$$

The separated variables for this system may be obtained from the proposed separated variables (3.9) on $\mathcal{E}_a$ using mapping (4.3).

5. Summary

A separation of variables for the generic Kowalevski–Goryachev–Chaplygin gyrostat is found. The separated variables are invariant with respect to the Abelian group of symplectic diffeomorphisms of the corresponding Lagrangian foliation and, therefore, belong to the invariant intersection of all the subfoliations. This invariance property allows us to calculate the separated variables explicitly.

Namely, the invariant separated variables lie on a symmetric product of two copies of single hyperelliptic curve and the separated coordinates are roots of the second-order equation with polynomial coefficients, so this separation stands a good chance of being quantized.

However, the proposed separated variables are complex functions of physical variables. It remains an open problem how to construct real solutions of the Kowalevski–Goryachev–Chaplygin gyrostat using complex hyperelliptic quadratures.

Acknowledgment

This work was partially supported by RFBR grant N02-01-00888.

References

[1] Kowalevski S V 1889 Sur le problème de la rotation d'un corps solide autour d'un point fixe *Acta Math.* **12** 177
Chaplygin S A 1897 O nekotorih sluchajah dvigenija tverdogo tela v gidkosti *Mat. Sbornik Krugka Ljubiteley Mat. Nauk* **20** 115–70
Chaplygin S A 1897 O nekotorih sluchajah dvigenija tverdogo tela v gidkosti *Mat. Sbornik Krugka Ljubiteley Mat. Nauk* **20** 173–246
Goryachev D N 1900 O dvigenii tjagelogo tverdogo tela vokrug nepodvignoj tochki *Mat. Sbornik Krugka Ljubiteley Mat. Nauk* **21** 431–8

[2] Chaplygin S A 903 Novoje chastnoye reshenije zadachi o dvigenii tverdogo tela v gidkosti *Trudy Otdeleniya Fiz. Nauk Imperatorskogo Obshchestva Lyubitelej Estestvoznaniya* **11** 7–10

[3] Kupershmidt B A and Vinogradov A M 1977 The structures of Hamiltonian mechanics *Russ. Math. Surv.* **32** 177–243

[4] Reyman A G and Semenov-Tian Shansky M A 1987 Lax representation with a spectral parameter for the Kowalewski top and its generalizations *Lett. Math. Phys.* **14** 55

[5] Sklyanin E K 1995 Separation of variables—new trends. Quantum field theory, integrable models and beyond (Kyoto, 1994) *Prog. Theor. Phys. Suppl.* **118** 35

[6] Kuznetsov V B 2002 Simultaneous separation for the Kowalewski and Goryachev–Chaplygin gyrostats *Preprint* nlin.SI/0201004

[7] Magnus K 1971 *Kreisel. Theorie und Anwendungen* (Berlin: Springer)
[8] Morosi C and Tondo G 2002 The quasi-bi-Hamiltonian formulation of the Lagrange top *J. Phys. A: Math. Gen.* **35** 1741
[9] Kuznetsov V B and Tsiganov A V 1989 A special case of Neumann's system and the Kowalewski–Chaplygin–Goryachev top *J. Phys. A: Math. Gen.* **22** L73
[10] Tsiganov A V 1997 The Kowalewski top, a new Lax representation *J. Math. Phys.* **38** 196
[11] Sokolov V V and Tsiganov A V 2001 Commutative Poisson subalgebras for the Sklyanin bracket and deformations of known integrable models *Theor. Math. Phys.* at press
(Sokolov V V and Tsiganov A V 2001 *Preprint* nlin.SI/0112011)
[12] Sklyanin E K 1988 Boundary conditions for integrable quantum systems *J. Phys. A: Math. Gen.* **21** 2375
[13] Belokolos E D, Bobenko A I, Enol'skij V Z, Its A R and Matveev V B 1994 *Algebro-Geometric Approach to Nonlinear Integrable Equations (Springer Series in Nonlinear Dynamics)* (Berlin: Springer)
[14] Tsiganov A V 2001 On the invariant separated variables *Regular Chaot. Dyn.* **6** 307
[15] Markushevich D 2001 Kowalewski top and genus-2 curves *J. Phys. A: Math. Gen.* **34** 2125